

NORM OF A DERIVATION ON A VON NEUMANN ALGEBRA⁽¹⁾

BY

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ABSTRACT. A derivation on an algebra $\mathfrak{A}$ is a linear function $\mathfrak{D}: \mathfrak{A} \rightarrow \mathfrak{A}$ satisfying $\mathfrak{D}(ab) = \mathfrak{D}(a)b + a\mathfrak{D}(b)$ for all a, b in $\mathfrak{A}$. If there exists an a in $\mathfrak{A}$ such that $\mathfrak{D}(b) = ab - ba$ for b in $\mathfrak{A}$, then $\mathfrak{D}$ is called the inner derivation induced by a . If $\mathfrak{A}$ is a von Neumann algebra, then by a theorem of Sakai [7], every derivation on $\mathfrak{A}$ is inner. In this paper we compute the norm of a derivation on a von Neumann algebra. Specifically we prove that if $\mathfrak{A}$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, T is in $\mathfrak{A}$, and $\mathfrak{D}_T$ is the derivation induced by T , then $\|\mathfrak{D}_T|_{\mathfrak{A}}\| = 2 \inf\{\|T - Z\|, Z \text{ in centre } \mathfrak{A}\}$.

In this paper we answer one of the questions raised by J. Stampfli in [8], viz. computing the norm of a derivation on an arbitrary C^* -algebra. It follows from [3, Lemma 3] that a derivation on a C^* -algebra acting on a Hilbert space $\mathcal{H}$ can be extended to its weak-closure in $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators acting on $\mathcal{H}$. Kaplansky's density theorem proves that this extension is achieved without increasing the norm. So there is no loss of generality in restricting our attention to von Neumann algebras. This result when T is selfadjoint has been proved by Kadison, Lance and Ringrose. But the methods have little in common.

1. Preliminaries. We shall use decomposition of a Hilbert space into a direct integral relative to a given abelian von Neumann algebra. We quote two theorems for future reference and refer the reader to [6] for notations and proofs.

Theorem. *To every commutative von Neumann algebra $\mathcal{C}$ on a separable Hilbert space $\mathcal{H}$ there corresponds a decomposition of $\mathcal{H}$ into direct integral with $\mathcal{C}$ as the totality of all operators of the form L_ϕ .*

Theorem. *Under the same hypothesis as in the previous theorem, for an arbitrary family $\mathfrak{A} \subseteq \mathcal{C}'$, every element A in $\mathfrak{A}$ is decomposable. Let $\mathfrak{A}(\lambda) = \{A(\lambda), A \text{ in } \mathfrak{A}\}$. Then the family $\mathfrak{A}(\lambda)$ is irreducible for almost all λ if and only if $\mathcal{C}$ is a maximal abelian subalgebra of $\mathfrak{A}'$.*

In [8], Stampfli has defined the centre of a bounded operator T as the unique

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complex number μ for which $\|T - \mu\| = \inf\{\|T - \lambda\|, \lambda \text{ a complex number}\}$. Following [8] we denote the centre of a bounded operator T by C_T , centre of an algebra $\mathfrak{A}$ by $\mathfrak{Z}(\mathfrak{A})$, and the complex numbers by $\mathbb{C}$. We assume that all Hilbert spaces are separable.

2. Lemma 1. *If $\mathfrak{A}$ is a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and if T is in $\mathfrak{A}$, then there exists Z_0 in $\mathfrak{Z}(\mathfrak{A})$ such that, for every projection P in $\mathfrak{Z}(\mathfrak{A})$, $\|(T - Z_0)P\| = \inf\{\|(T - Z)P\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$.*

Proof. First we note that if $\alpha = \inf\{\|T - Z\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$, then there exists a sequence $\{Z_n\} \subseteq \mathfrak{Z}(\mathfrak{A})$ such that $\|Z_n - T\| \rightarrow \alpha$. Also, $\|Z_n\| \leq \|T - Z_n\| + \|T\|$ implies that $\{\|Z_n\|\}$ is a bounded sequence. Hence there exists a subsequence $\{Z_{n_k}\}$ such that $Z_{n_k} \rightarrow Z_0$ in the weak operator topology. It follows that $Z_0 \in \mathfrak{Z}(\mathfrak{A})$ and further,

$$\|T - Z_0\| \leq \liminf \|T - Z_{n_k}\| = \alpha.$$

Thus $\|T - Z_0\| = \alpha$.

Now if P is a projection in $\mathfrak{Z}(\mathfrak{A})$, applying the previous case to the algebra $\mathfrak{A}P$ and operator TP , it follows that there exists $Z_P \in \mathfrak{Z}(\mathfrak{A})$ such that $\|(T - Z_P)P\| = \inf\{\|(T - Z)P\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$.

If $\{P_i\}_{i=1}^n$ is a family of pairwise orthogonal projections in $\mathfrak{Z}(\mathfrak{A})$ with $\sum_{i=1}^n P_i = I$, we know that there exist $\{Z_i\}_{i=1}^n \subseteq \mathfrak{Z}(\mathfrak{A})$ satisfying $\|(T - Z_i)P_i\| = \inf\{\|(T - Z)P_i\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$, $1 \leq i \leq n$. Define $Z^1 = \sum_{i=1}^n Z_i P_i$ and denote $Z^1 = Z_{\{P_1, \dots, P_n\}}$. Then $\|(T - Z^1)P_i\| = \inf\{\|(T - Z)P_i\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$ for $1 \leq i \leq n$.

Let $\mathcal{F}$ be the class of all finite orthogonal families $\{Q_1, \dots, Q_n\}$ with sum I . We say $\{P_1, \dots, P_m\} \leq \{Q_1, \dots, Q_n\}$ if each P_i is a sum of a subset of Q 's. Note that if $\{Q_1, \dots, Q_n\} \in \mathcal{F}$, then

$$\|T - Z_{\{Q_1, \dots, Q_n\}}\| = \inf\{\|T - Z\|, Z \text{ in } \mathfrak{Z}(\mathfrak{A})\}.$$

It follows that $\{Z_{\{Q_1, \dots, Q_n\}}, \{Q_1, \dots, Q_n\} \in \mathcal{F}\}$ is a bounded net. Hence there is a cofinal subset which converges to Z_0 in $\mathfrak{Z}(\mathfrak{A})$ in weak-operator-topology. We now prove that, for any projection Q in $\mathfrak{Z}(\mathfrak{A})$,

$$\|(T - Z_0)Q\| = \inf\{\|(T - Z)Q\|, Z \text{ in } \mathfrak{Z}(\mathfrak{A})\}.$$

For if Q is a projection in $\mathfrak{Z}(\mathfrak{A})$, $\{Q, I - Q\} \in \mathcal{F}$, and if $\{Q_1, \dots, Q_n\} \geq \{I - Q, Q\}$, we may write (with possible renumbering) $Q = Q_1 + \dots + Q_k$, $k \leq n$.

$$\begin{aligned} \|(T - Z_{\{Q_1, \dots, Q_n\}})Q\| &= \sup_{1 \leq i \leq k} \|(T - Z_{\{Q_1, \dots, Q_n\}})Q_i\| \\ &\leq \sup_{1 \leq i \leq k} \|(T - Z)Q_i\| \quad \text{for all } Z \text{ in } \mathfrak{Z}(\mathfrak{A}). \end{aligned}$$

Thus $\|(T - Z_{\{Q_1, \dots, Q_n\}})Q\| \leq \|(T - Z)Q\|$ for all $Z \in \mathcal{Z}(\mathfrak{U})$ and $\{Q_1, \dots, Q_n\} \geq \{Q, I - Q\}$.

Since weak limits do not increase norm,

$$\|(T - Z_0)Q\| \leq \|(T - Z)Q\| \quad \text{for all } Z \text{ in } \mathcal{Z}(\mathfrak{U}).$$

Lemma 2. Let $\mathcal{H} = \int^\oplus \mathcal{H}_\lambda d\sigma(\lambda)$ and T in $\mathcal{B}(\mathcal{H})$ be a decomposable operator, i.e. $T = T(\lambda)$ where $\{T(\lambda)\}$ is an essentially bounded, measurable operator function. Then the scalar-valued function $C_{T(\lambda)}$ is essentially bounded and measurable.

Proof. Let $\mathcal{H} = \int^\oplus \mathcal{H}_\lambda d\sigma(\lambda)$ and let T in $\mathcal{B}(\mathcal{H})$ be written as $T = \{T(\lambda)\}$. Let $\mathcal{Z} = \{L_\phi, \phi \in L^\infty(\Lambda, \sigma)\}$ and $\mathfrak{U} = \mathcal{Z}' =$ the algebra of all decomposable operators on $\mathcal{H}$.

Then $T \in \mathfrak{U}$ and $\mathcal{Z}(\mathfrak{U}) = \mathfrak{U} \cap \mathfrak{U}' = \mathcal{Z}' \cap \mathcal{Z} = \mathcal{Z}$. By Lemma 1, there exists $Z_0 = L_\phi$ in $\mathcal{Z}$ such that, for every projection P in $\mathcal{Z}$,

$$\|(T - Z_0)P\| = \inf \{ \|(T - Z)P\|, Z \in \mathcal{Z} \}.$$

We assert that $\phi(\lambda) = C_{T(\lambda)}$ a.e.

If $\mu \in \mathbb{C}$ and $\epsilon > 0$, let

$$E = \{ \lambda \in \Lambda, \|T(\lambda) - \mu I_\lambda\| \leq \|T(\lambda) - \phi(\lambda)I_\lambda\| - \epsilon \}$$

and ψ_E denote the characteristic function of E . Then $Z_1 = \{\mu I_\lambda\} \in \mathcal{Z}$ and $P = \{\psi_E(\lambda)I_\lambda\}$ is a projection in $\mathcal{Z}$. Since $\|(T - Z_0)P\| \leq \|(T - Z_1)P\|$, it follows that $\sigma(E) = 0$. If $\{\mu_n\}_1^\infty$ is a dense subset of complex numbers, write

$$F = \bigcup_{n=1}^{\infty} \{ \lambda \in \Lambda, \|T(\lambda) - \mu_n I_\lambda\| < \|T(\lambda) - \phi(\lambda)I_\lambda\| \}.$$

It follows that $\sigma(F) = 0$.

If $\lambda \notin F$ and $\mu \in \mathbb{C}$, $\mu_{n_k} \rightarrow \mu$ for some subsequence $\{n_k\}$, and

$$\|T(\lambda) - \mu I_\lambda\| = \lim_{k \rightarrow \infty} \|T(\lambda) - \mu_{n_k} I_\lambda\| \geq \|T(\lambda) - \phi(\lambda)I_\lambda\|.$$

By uniqueness of $C_{T(\lambda)}$, it follows that, for $\lambda \notin F$, $C_{T(\lambda)} = \phi(\lambda)$. Since $\sigma(F) = 0$, this proves the claim and consequently the lemma.

Corollary 1. The element Z obtained in Lemma 1 is unique, when $\mathfrak{U}$ is a type I algebra.

Theorem 1. Let $\mathfrak{U}$ be a von Neumann algebra on $\mathcal{H}$ and assume that $\mathfrak{U}'$ is abelian. Then for T in $\mathfrak{U}$, there exists Z_0 in $\mathcal{Z}(\mathfrak{U}) = \mathfrak{U}'$ such that $\|\mathcal{D}_T|_{\mathfrak{U}}\| = 2\|T - Z_0\|$.

Proof. Let $\mathcal{H} = \int^\oplus \mathcal{H}_\lambda d\sigma(\lambda)$ be the decomposition corresponding to the

abelian algebra $\mathcal{Z}(\mathfrak{U}) = \mathfrak{U}'$. Then $\mathcal{Z}(\mathfrak{U}) = \{L_\phi, \phi \in L^\infty(\Lambda, \sigma)\}$ and $\mathfrak{U}$ = the family of all decomposable operators on $\mathcal{H}$. Also there exists a σ -null set F such that $\lambda \notin F$ implies $\mathfrak{U}(\lambda)$ is an irreducible algebra. Define $\phi(\lambda) = C_{T(\lambda)}$ if $\lambda \notin F$ and $\phi(\lambda) = 0$ if $\lambda \in F$. Then by Lemma 2, $\phi \in L^\infty(\Lambda, \sigma)$ and hence $L_\phi \in \mathcal{Z}(\mathfrak{U})$. By Stampfli's theorem [8], for $\lambda \notin F$, $\|\mathfrak{D}_{T(\lambda)}| \mathfrak{U}(\lambda)\| = 2\|T(\lambda) - C_{T(\lambda)}I_\lambda\|$ and thus

$$2\|T - L_\phi\| = 2 \operatorname{ess\,sup}_\lambda \|T(\lambda) - C_{T(\lambda)}I_\lambda\| = \operatorname{ess\,sup}_\lambda \|\mathfrak{D}_{T(\lambda)}| \mathfrak{U}(\lambda)\|.$$

We prove that $\operatorname{ess\,sup}_\lambda \|\mathfrak{D}_{T(\lambda)}| \mathfrak{U}(\lambda)\| \leq \|\mathfrak{D}_T| \mathfrak{U}\|$.

We set $\Lambda_k = \{\lambda \in \Lambda, \dim \mathcal{H}_\lambda = k\}$. Then $\Lambda = \bigcup_k \Lambda_k$ and it is sufficient to prove that

$$\sigma\{\lambda \in \Lambda_k, \|\mathfrak{D}_{T(\lambda)}| \mathfrak{U}(\lambda)\| > \|\mathfrak{D}_T| \mathfrak{U}\|\} = \emptyset \quad \text{for all } k.$$

Fix k for which $\sigma(\Lambda_k) \neq 0$ and write $\mathcal{H}_k = \int_{\Lambda_k}^\oplus \mathcal{H}_\lambda d\sigma(\lambda)$ where all $\mathcal{H}_\lambda$ can be identified with the same Hilbert space $\mathcal{K}$. Let $\{B_n, n \geq 1\}$ be a sequence which is weakly dense in the unit ball of $\mathcal{B}(\mathcal{K})$. Defining $A_n(\lambda) = B_n$ for $\lambda \in \Lambda_k$ and $A_n(\lambda) = 0$ for $\lambda \notin \Lambda_k$, we see that $A_n = A_n(\lambda)$ is an essentially bounded, measurable, operator function and hence $A_n \in \mathcal{B}(\mathcal{H})$ with $\|A_n\| \leq 1$. Since by definition A_n is a decomposable operator, $A_n \in \mathfrak{U}$. Now

$$\begin{aligned} \operatorname{ess\,sup}_{\lambda \in \Lambda_k} \|B_n T(\lambda) - T(\lambda) B_n\| &= \operatorname{ess\,sup}_{\lambda \in \Lambda} \|A_n(\lambda) T(\lambda) - T(\lambda) A_n(\lambda)\| \\ &= \|A_n T - T A_n\| \leq \|\mathfrak{D}_T| \mathfrak{U}\|. \end{aligned}$$

Hence if we write $E_n = \{\lambda \in \Lambda_k, \|T(\lambda) B_n - B_n T(\lambda)\| > \|\mathfrak{D}_T| \mathfrak{U}\|\}$ and $E = \bigcup_{n=1}^\infty E_n$, we see that $\sigma(E_n) = 0$ and hence $\sigma(E) = 0$.

Next, we will show that, for $\lambda \in \Lambda_k$, $\lambda \notin E$,

$$\|\mathfrak{D}_{T(\lambda)}| \mathfrak{U}(\lambda)\| \leq \|\mathfrak{D}_T| \mathfrak{U}\|.$$

Fix $\lambda_0 \in \Lambda_k$, $\lambda_0 \notin E$ and let $A \in \mathfrak{U}$ with $\|A(\lambda_0)\| \leq 1$. Then there exists a subsequence $\{B_{n_k}\}$ (depending on λ_0), such that $B_{n_k} \rightarrow A(\lambda_0)$ in weak-operator-topology. Hence

$$\|\mathfrak{D}_T(A)(\lambda_0)\| \leq \liminf \|B_{n_k} T(\lambda_0) - T(\lambda_0) B_{n_k}\| \leq \|\mathfrak{D}_T| \mathfrak{U}\|.$$

Since $\mathfrak{D}_T(A)(\lambda_0) = \mathfrak{D}_{T(\lambda_0)} A(\lambda_0)$ we have proved that

$$\|\mathfrak{D}_{T(\lambda)}| \mathfrak{U}(\lambda)\| \leq \|\mathfrak{D}_T| \mathfrak{U}\| \quad \text{for all } \lambda \in \Lambda_k, \lambda \notin E.$$

Thus $\operatorname{ess\,sup}_\lambda \|\mathfrak{D}_{T(\lambda)}| \mathfrak{U}(\lambda)\| \leq \|\mathfrak{D}_T| \mathfrak{U}\|$. Hence

$$2\|T - L_\phi\| = \operatorname{ess\,sup}_\lambda \|\mathfrak{D}_{T(\lambda)}| \mathfrak{U}(\lambda)\| = \|\mathfrak{D}_T| \mathfrak{U}\| \leq 2\|T - L_\phi\|$$

and

$$2 \|T - L_\phi\| = \|\mathcal{D}_T | \mathfrak{U}\|.$$

Corollary 2. *If $\mathfrak{U}$ is a von Neumann algebra with $\mathfrak{U}'$ abelian and T is in $\mathfrak{U}$, then there exists Z_0 in $\mathcal{Z}(\mathfrak{U})$ such that, for every projection P in $\mathcal{Z}(\mathfrak{U})$,*

$$\|\mathcal{D}_{TP} | \mathfrak{U}P\| = 2 \|(T - Z_0)P\|.$$

Proof. Let P be a projection in $\mathcal{Z}(\mathfrak{U}) = \mathfrak{U}'$. Consider the algebra $\mathfrak{U}P$ and $S = TP$. Since $(\mathfrak{U}P)' = \mathfrak{U}'P$, $(\mathfrak{U}P)'$ is abelian.

Since P is a projection in $\mathcal{Z}(\mathfrak{U})$, $P = L_\eta$, where η is the characteristic function of a measurable set E . By Theorem 1, we have

$$\|\mathcal{D}_S | \mathfrak{U}P\| = 2 \|S - L_\phi\| \quad \text{where } \psi(\lambda) = C_{S(\lambda)} \text{ a.e.}$$

It follows that $C_{S(\lambda)} = \eta(\lambda)C_{T(\lambda)}$ and hence

$$\|\mathcal{D}_S | \mathfrak{U}P\| = 2 \|S - L_\psi\| = 2 \operatorname{ess\,sup}_\lambda \|(T(\lambda) - C_{T(\lambda)})\eta(\lambda)\| = 2 \|(T - Z_0)P\|.$$

Theorem 2. *Let $\mathfrak{U}$ be a von Neumann algebra on $\mathcal{H}$, and let T be in $\mathfrak{U}$. Then there exists ξ in $\mathfrak{U}'$ such that for every projection P in $\mathcal{Z}(\mathfrak{U})$, $\|\mathcal{D}_{TP} | \mathfrak{U}P\| = 2 \|(T - \xi)P\|$.*

Proof. Let $\mathfrak{U}_1$ be the type I algebra containing $\mathfrak{U}$, as constructed in [3, p. 283]. Then by Kadison's argument, $\|\mathcal{D}_T | \mathfrak{U}\| = \|\mathcal{D}_T | \mathfrak{U}_1\|$. Moreover, applying the same argument to $\mathfrak{U}P$ in place of $\mathfrak{U}$, we see that, for any central projection P in $\mathfrak{U}$, $\|\mathcal{D}_{TP} | \mathfrak{U}P\| = \|\mathcal{D}_{TP} | \mathfrak{U}_1P\|$. By Theorem 1, there exists an element ξ in $\mathcal{Z}(\mathfrak{U}_1)$ such that $\|\mathcal{D}_{TQ} | \mathfrak{U}_1Q\| = 2 \|(T - \xi)Q\|$ for every central projection Q in $\mathfrak{U}$. Since $\mathcal{Z}(\mathfrak{U}) \subseteq \mathcal{Z}(\mathfrak{U}_1) \subseteq \mathfrak{U}'$, this completes the proof.

Theorem 3. *Let $\mathfrak{U}$ be a von Neumann algebra on a separable Hilbert space and let T be in $\mathfrak{U}$. Then there exists a unique Z_0 in $\mathcal{Z}(\mathfrak{U})$ such that, for every projection P in $\mathcal{Z}(\mathfrak{U})$, $\|\mathcal{D}_{TP} | \mathfrak{U}P\| = 2 \|(T - Z_0)P\|$.*

Proof. By Theorem 2, there exists ξ in $\mathfrak{U}'$ such that, for every projection P in $\mathcal{Z}(\mathfrak{U})$, $\|\mathcal{D}_{TP} | \mathfrak{U}P\| = 2 \|(T - \xi)P\|$. Let $K_\xi = \{\sum_{i=1}^n \lambda_i \mathfrak{U}_i \xi \mathfrak{U}_i^*, \lambda_i \geq 0, \sum \lambda_i = 1, \mathfrak{U}_i \text{ unitary in } \mathfrak{U}'\}$, and $K_\xi^=$, the norm closure of K_ξ . If ξ' is in K_ξ , and P is a projection in $\mathcal{Z}(\mathfrak{U})$,

$$(T - \xi')P = \sum_{i=1}^n \lambda_i (T - \mathfrak{U}_i^* \xi \mathfrak{U}_i)P = \sum_{i=1}^n \lambda_i \mathfrak{U}_i^* (TP - \xi P) \mathfrak{U}_i.$$

Hence $\|(T - \xi')P\| \leq \|(T - \xi)P\|$.

Since $\|\mathcal{D}_{TP} | \mathfrak{U}P\| \leq 2 \|(T - \xi')P\| \leq 2 \|(T - \xi)P\| = \|\mathcal{D}_{TP} | \mathfrak{U}P\|$, we have $\|\mathcal{D}_{TP} | \mathfrak{U}P\| = 2 \|(T - \xi')P\|$ for all $\xi' \in K_\xi$.

By continuity, the same equality holds for all $\xi' \in K_\xi^=$. By [1, Theorem 1,

p. 272], $K_{\xi}^{\perp} \cap \mathcal{Z}(\mathcal{U}') \neq \emptyset$. Hence if Z_0 in $K_{\xi}^{\perp} \cap \mathcal{Z}(\mathcal{U})$, $2\|(T - Z_0)P\| = \|\mathcal{D}_{TP} | \mathcal{U}P\|$, for every central projection P of $\mathcal{U}$.

Now we prove uniqueness of Z_0 . If $\mathcal{U}_1$ is the type I algebra containing $\mathcal{U}$ as before, Q is a projection in $\mathcal{Z}(\mathcal{U}_1) \subseteq \mathcal{U}'$, then we assert that $\|\mathcal{D}_{TQ} | \mathcal{U}_1Q\| = 2\|(T - Z_0)Q\|$.

Let P be the central support of Q in $\mathcal{Z}(\mathcal{U})$. Then for every A in $\mathcal{U}$, $\|AP\| = \|AQ\|$ [2].

Now,

$$\|\mathcal{D}_{TP} | \mathcal{U}P\| = 2\|(T - Z_0)P\| = 2\|(T - Z_0)Q\| \geq \|\mathcal{D}_{TQ} | \mathcal{U}_1Q\|.$$

On the other hand, if X is in $\mathcal{U}$,

$$\begin{aligned} \|\mathcal{D}_T(X)P\| &= \|\mathcal{D}_{TP}(XP)\| = \|\mathcal{D}_T(X)Q\| = \|\mathcal{D}_{TQ}(XQ)\| \\ &\leq \|\mathcal{D}_{TQ} | \mathcal{U}_1Q\| \|XQ\| = \|\mathcal{D}_{TQ} | \mathcal{U}_1Q\| \|XP\|. \end{aligned}$$

Now Corollary 1 proves the uniqueness of Z_0 .

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